Section A: Pure Mathematics

1 Let P be a given point on a given curve C. The *osculating circle* to C at P is defined to be the circle that satisfies the following two conditions at P: it touches C; and the rate of change of its gradient is equal to the rate of change of the gradient of C.

Find the centre and radius of the osculating circle to the curve $y = 1 - x + \tan x$ at the point on the curve with x-coordinate $\frac{1}{4}\pi$.

2 Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x\,.$$

Find and prove a similar result for $\sin 3x$ in terms of $\sin x$.

(i) Let

$$\mathrm{I}(\alpha) = \int_0^\alpha \big(7\sin x - 8\sin^3 x\big)\mathrm{d}x\,.$$

Show that

$$\mathrm{I}(\alpha) = -\tfrac{8}{3}c^3 + c + \tfrac{5}{3}\,,$$

where $c = \cos\alpha$. Write down one value of c for which $\mathrm{I}(\alpha) = 0$.

(ii) Useless Eustace believes that

$$\int \sin^n x\,\mathrm{d}x = \frac{\sin^{n+1} x}{n+1}$$

for $n = 1,\ 2,\ 3, \ldots$. Show that Eustace would obtain the correct value of $\mathrm{I}(\beta)$, where $\cos\beta = -\frac{1}{6}$.

Find all values of α for which he would obtain the correct value of $\mathrm{I}(\alpha)$.

3 The first four terms of a sequence are given by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_3 = 2$. The general term is given by

$$F_n = a\lambda^n + b\mu^n\,, \qquad (*)$$

where a, b, λ and μ are independent of n, and a is positive.

(i) Show that $\lambda^2 + \lambda\mu + \mu^2 = 2$, and find the values of λ, μ, a and b.

(ii) Use $(*)$ to evaluate F_6.

(iii) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}}$.

4 **(i)** Let

$$I = \int_0^a \frac{\mathrm{f}(x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x\,.$$

Use a substitution to show that

$$I = \int_0^a \frac{\mathrm{f}(a-x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x$$

and hence evaluate I in terms of a.

Use this result to evaluate the integrals

$$\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(x+\frac{\pi}{4})}\,\mathrm{d}x\,.$$

(ii) Evaluate

$$\int_{\frac{1}{2}}^2 \frac{\sin x}{x\big(\sin x + \sin\frac{1}{x}\big)}\,\mathrm{d}x\,.$$

5 The points A and B have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $5\mathbf{i}-\mathbf{j}-\mathbf{k}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

(i) The line L_1 has equation $\mathbf{r} = \lambda(m\mathbf{i} + n\mathbf{j} + p\mathbf{k})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r} = \mu(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha = \angle AOB$, determine a relationship between u, v and w.

Hence describe the surface with Cartesian equation $x^2 + y^2 + z^2 = 2(yz + zx + xy)$.

6 Each edge of the tetrahedron $ABCD$ has unit length. The face ABC is horizontal, and P is the point in ABC that is vertically below D.

(i) Find the length of PD.

(ii) Show that the cosine of the angle between adjacent faces of the tetrahedron is $1/3$.

(iii) Find the radius of the largest sphere that can fit inside the tetrahedron.

7 **(i)** By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1+q)\,,$$

where $q > 0$ and $q \neq 1$, crosses the x-axis once only.

(ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1+q) = 0\,,$$

and that

$$x = u + q/u\,,$$

obtain a quadratic equation satisfied by u^3. Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

(iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β. Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp\,.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q. Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q.

8 The curves C_1 and C_2 are defined by

$$y = \mathrm{e}^{-x} \quad (x > 0) \quad \text{and} \quad y = \mathrm{e}^{-x}\sin x \quad (x > 0),$$

respectively. Sketch roughly C_1 and C_2 on the same diagram.

Let x_n denote the x-coordinate of the nth point of contact between the two curves, where $0 < x_1 < x_2 < \cdots$, and let A_n denote the area of the region enclosed by the two curves between x_n and x_{n+1}. Show that

$$A_n = \tfrac{1}{2}(\mathrm{e}^{2\pi} - 1)\mathrm{e}^{-(4n+1)\pi/2}$$

and hence find $\displaystyle\sum_{n=1}^{\infty} A_n$.

Section B: Mechanics

9 Two points A and B lie on horizontal ground. A particle P_1 is projected from A towards B at an acute angle of elevation α and simultaneously a particle P_2 is projected from B towards A at an acute angle of elevation β. Given that the two particles collide in the air a horizontal distance b from B, and that the collision occurs after P_1 has attained its maximum height h, show that

$$2h \cot \beta < b < 4h \cot \beta$$

and

$$2h \cot \alpha < a < 4h \cot \alpha \,,$$

where a is the horizontal distance from A to the point of collision.

10 **(i)** In an experiment, a particle A of mass m is at rest on a smooth horizontal table. A particle B of mass bm, where $b > 1$, is projected along the table directly towards A with speed u. The collision is perfectly elastic.

Find an expression for the speed of A after the collision in terms of b and u, and show that, irrespective of the relative masses of the particles, A cannot be made to move at twice the initial speed of B.

(ii) In a second experiment, a particle B_1 is projected along the table directly towards A with speed u. This time, particles B_2, B_3, $\ldots$, B_n are at rest in order on the line between B_1 and A. The mass of B_i $(i = 1, 2, \ldots, n)$ is $\lambda^{n+1-i}m$, where $\lambda > 1$. All collisions are perfectly elastic. Show that, by choosing n sufficiently large, there is no upper limit on the speed at which A can be made to move.

In the case $\lambda = 4$, determine the least value of n for which A moves at more than $20u$. You may use the approximation $\log_{10} 2 \approx 0.30103$.

11 A uniform rod AB of length $4L$ and weight W is inclined at an angle θ to the horizontal. Its lower end A rests on a fixed support and the rod is held in equilibrium by a string attached to the rod at a point C which is $3L$ from A. The reaction of the support on the rod acts in a direction α to AC and the string is inclined at an angle β to CA. Show that

$$\cot \alpha = 3 \tan \theta + 2 \cot \beta \,.$$

Given that $\theta = 30°$ and $\beta = 45°$, show that $\alpha = 15°$.

Section C: Probability and Statistics

12 The continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} a & \text{for } 0 \leqslant x < k \\ b & \text{for } k \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$\mathrm{E}(X) = \frac{1 - 2b + ab}{2(a-b)}\,.$$

(ii) Show that the median, M, of X is given by $M = \dfrac{1}{2a}$ if $a + b \geqslant 2ab$ and obtain an expression for the median if $a + b \leqslant 2ab$.

(iii) Show that $M < \mathrm{E}(X)$.

13 Rosalind wants to join the Stepney Chess Club. In order to be accepted, she must play a challenge match consisting of several games against Pardeep (the Club champion) and Quentin (the Club secretary), in which she must win at least one game against each of Pardeep and Quentin. From past experience, she knows that the probability of her winning a single game against Pardeep is p and the probability of her winning a single game against Quentin is q, where $0 < p < q < 1$.

(i) The challenge match consists of three games. Before the match begins, Rosalind must choose either to play Pardeep twice and Quentin once or to play Quentin twice and Pardeep once. Show that she should choose to play Pardeep twice.

(ii) In order to ease the entry requirements, it is decided instead that the challenge match will consist of four games. Now, before the match begins, Rosalind must choose whether to play Pardeep three times and Quentin once (strategy 1), or to play Pardeep twice and Quentin twice (strategy 2) or to play Pardeep once and Quentin three times (strategy 3).

Show that, if $q - p > \frac{1}{2}$, Rosalind should choose strategy 1.

If $q - p < \frac{1}{2}$, give examples of values of p and q to show that strategy 2 can be better or worse than strategy 1.