Section A: Pure Mathematics

1 In the game of "Colonel Blotto" there are two players, Adam and Betty. First Adam chooses three non-negative integers a_1, a_2 and a_3, such that $a_1 + a_2 + a_3 = 9$, and then Betty chooses non-negative integers b_1, b_2 and b_3, such that $b_1 + b_2 + b_3 = 9$. If $a_1 > b_1$ then Adam scores one point; if $a_1 < b_1$ then Betty scores one point; and if $a_1 = b_1$ no points are scored. Similarly for a_2, b_2 and a_3, b_3. The winner is the player who scores the greater number of points: if the socres are equal then the game is drawn. Show that, if Betty knows the numbers a_1, a_2 and a_3, she can always choose her numbers so that she wins. Show that Adam can choose a_1, a_2 and a_3 in such a way that he will never win no matter what Betty does.

Now suppose that Adam is allowed to write down two triples of numbers and that Adam wins unless Betty can find one triple that beats both of Adam's choices (knowing what they are). Confirm that Adam wins by writing down $(5, 3, 1)$ and $(3, 1, 5)$.

2 **(i)** Evaluate

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,\mathrm{d}x,$$

where m, n are integers, taking into account any special cases that arise.

(ii) Find $\displaystyle\int \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x$.

3 **(i)** Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y - 3y^2 = -2$$

by making the substitution $y = -\dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

(ii) Solve the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^2y^2 = 1$$

by making the substitution

$$y = \frac{1}{x} + \frac{1}{v},$$

where v is a function of x.

4 Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2,1,-1) \qquad \mathbf{p}_2 = (1,0,-2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4,3,0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0,-3,1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2\,|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2)\cdot(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

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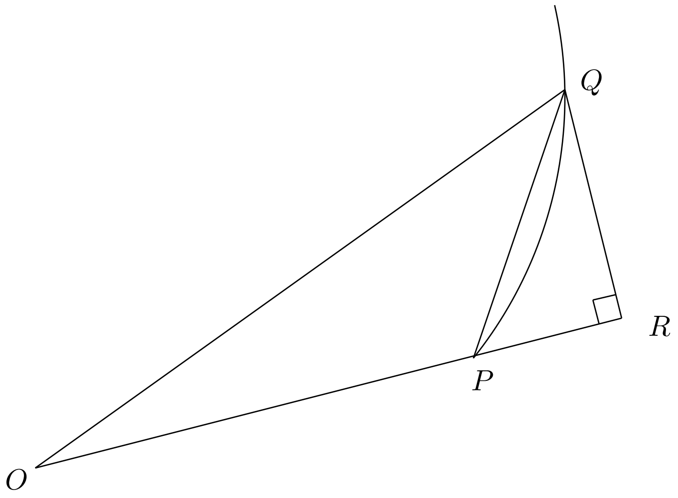

In the diagram, O is the origin, P is a point of a curve $r = r(\theta)$ with coordinates (r, θ) and Q is another point of the curve, close to P, with coordinates $(r + \delta r, \theta + \delta\theta)$. The angle $\angle PRQ$ is a right angle. By calculating $\tan \angle QPR$, show that the angle at which the curve cuts OP is

$$\tan^{-1}\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right).$$

Let α be a constant angle, $0 < \alpha < \frac{1}{2}\pi$. The curve with the equation

$$r = \mathrm{e}^{\theta \cot \alpha}$$

in polar coordinates is called an *equiangular spiral*. Show that it cuts every radius line at an angle α. Sketch the spiral.

Find the length of the complete turn of the spiral beginning at $r = 1$ and going outwards. What is the total length of the part of the spiral for which $r \leqslant 1$?

[You may assume that the arc length s of the curve satisfies

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2.]$$

6 *In this question,* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{X}$ *are non-zero* 2×2 *real matrices.*

Are the following assertions true or false? You must provide a proof or a counterexample in each case.

(i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.

(ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.

(iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

7 The integers a, b and c satisfy

$$2a^2 + b^2 = 5c^2.$$

By considering the possible values of $a \pmod 5$ and $b \pmod 5$, show that a and b must both be divisible by 5.

By considering how many times a, b and c can be divided by 5, show that the only solution is $a = b = c = 0$.

8 Suppose that $a_i > 0$ for all $i > 0$. Show that

$$a_1 a_2 \leqslant \left(\frac{a_1 + a_2}{2} \right)^2 .$$

Prove by induction that for all positive integers m

$$a_1 \cdots a_{2^m} \leqslant \left(\frac{a_1 + \cdots + a_{2^m}}{2^m} \right)^{2^m} . \qquad (*)$$

If $n < 2^m$, put $b_1 = a_2, b_2 = a_2, \cdots, b_n = a_n$ and $b_{n+1} = \cdots = b_{2^m} = A$, where

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

By applying $(*)$ to the b_i, show that

$$a_1 \cdots a_n A^{(2^m - n)} \leqslant A^{2^m}$$

(notice that $b_1 + \cdots + b_n = nA$). Deduce the (arithmetic mean)/(geometric mean) inequality

$$(a_1 \cdots a_n)^{1/n} \leqslant \frac{a_1 + \cdots + a_n}{n}.$$

9 *In this question, the argument of a complex number is chosen to satisfy* $0 \leqslant \arg z < 2\pi$.

Let z be a complex number whose imaginary part is positive. What can you say about $\arg z$?

The complex numbers z_1, z_2 and z_3 all have positive imaginary part and $\arg z_1 < \arg z_2 < \arg z_3$. Draw a diagram that shows why

$$\arg z_1 < \arg(z_1 + z_2 + z_3) < \arg z_3.$$

Prove that $\arg(z_1 z_2 z_3)$ is never equal to $\arg(z_1 + z_2 + z_3)$.

10 Verify that if

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} -1 & 8 \\ 8 & 11 \end{pmatrix}$$

then $\mathbf{PAP}$ is a diagonal matrix.

Put $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. By writing

$$\mathbf{x} = \mathbf{P}\mathbf{x}_1 + \mathbf{a}$$

for a suitable vector $\mathbf{a}$, show that the equation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} - 11 = 0,$$

where $\mathbf{b} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, becomes

$$3x_1^2 - y_1^2 = c$$

for some constant c (which you should find).

Section B: Mechanics

11 *In this question, take the value of g to be* $10\ \mathrm{ms}^{-2}$.

A body of mass m kg is dropped vertically into a deep pool of liquid. Once in the liquid, it is subject to gravity, an upward buoyancy force of $\frac{6}{5}$ times its weight, and a resistive force of $2mv^2\mathrm{N}$ opposite to its direction of travel when it is travelling at speed $v\ \mathrm{ms}^{-1}$. Show that the body stops sinking less than $\frac{1}{4}\pi$ seconds after it enters the pool.

Suppose now that the body enters the liquid with speed $1\ \mathrm{ms}^{-1}$. Show that the body descends to a depth of $\frac{1}{4}\ln 2$ metres and that it returns to the surface with speed $\frac{1}{\sqrt{2}}\ \mathrm{ms}^{-1}$, at a time

$$\frac{\pi}{8} + \frac{1}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

seconds after entering the pool.

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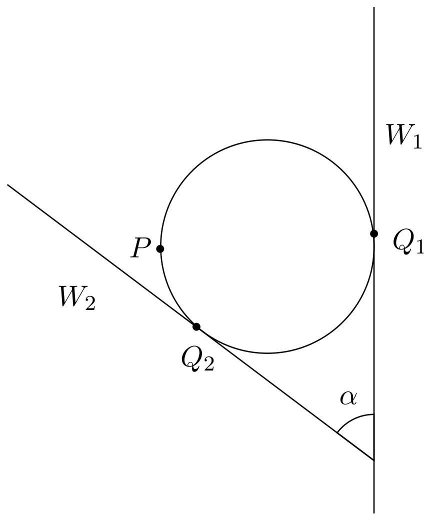

A uniform sphere of mass M and radius r rests between a vertical wall W_1 and an inclined plane W_2 that meets W_1 at an angle α. Q_1 and Q_2 are the points of contact of the sphere with W_1 and W_2 resectively, as shown in the diagram. A particle of mass m is attached to the sphere at P, where PQ_1 is a diameter, and the system is released. The sphere is on the point of slipping at Q_1 and at Q_2. Show that if the coefficients of friction between the sphere and W_1 and W_2 are μ_1 and μ_2 respectively, then

$$m = \frac{\mu_2 + \mu_1 \cos\alpha - \mu_1\mu_2 \sin\alpha}{(2\mu_1\mu_2 + 1)\sin\alpha + (\mu_2 - 2\mu_1)\cos\alpha - \mu_2} M.$$

If the sphere is on the point of rolling about Q_2 instead of slipping, show that

$$m = \frac{M}{\sec\alpha - 1}.$$

13 The force F of repulsion between two particles with positive charges Q and Q' is given by $F = kQQ'/r^2$, where k is a positive constant and r is the distance between the particles. Two small beads P_1 and P_2 are fixed to a straight horizontal smooth wire, a distance d apart. A third bead P_3 of mass m is free to move along the wire between P_1 and P_3. The beads carry positive electrical charges Q_1, Q_2 and Q_3. If P_3 is in equilibrium at a distance a from P_1, show that

$$a = \frac{d\sqrt{Q_1}}{\sqrt{Q_1} + \sqrt{Q_2}}.$$

Suppose that P_3 is displaced slightly from its equilibrium position and released from rest. Show that it performs approximate simple harmonic motion with period

$$\frac{\pi d}{(\sqrt{Q_1} + \sqrt{Q_2})^2} \sqrt{\frac{2md\sqrt{Q_1 Q_2}}{kQ_3}}.$$

[You may use the fact that $\dfrac{1}{(a+y)^2} \approx \dfrac{1}{a^2} - \dfrac{2y}{a^3}$ for small y.]

14 A ball of mass m is thrown vertically upwards from the floor of a room of height h with speed $\sqrt{2kgh}$, where $k > 1$. The coefficient of restitution between the ball and the ceiling or floor is a. Both the ceiling and floor are level. Show that the kinetic energy of the ball immediately before hitting the ceiling for the nth time is

$$mgh\left(a^{4n-4}(k-1) + \frac{a^{4n-4}-1}{a^2+1}\right).$$

Hence show that the number of times the ball hits the ceiling is at most

$$1 - \frac{\ln[a^2(k-1)+k]}{4\ln a}.$$

Section C: Probability and Statistics

15 Two computers, LEP and VOZ are programmed to add numbers after first approximating each number by an integer. LEP approximates the numbers by rounding: that is, it replaces each number by the nearest integer. VOZ approximates by truncation: that is, it replaces each number by the largest integer less than or equal to the number. The fractional parts of the numbers to be added are uniformly and independently distributed. (The fractional part of a number a is $a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ is the largest integer less than or equal to a.) Both computers approximate and add 1500 numbers. For each computer, find the probability that the magnitude of error in the answer will exceed 15.

How many additions can LEP perform before the probability that the magnitude of error is less than 10 drops below 0.9?

16 At the terminus of a bus route, passengers arrive at an average rate of 4 per minute according to a Poisson process. Each minute, on the minute, one bus arrives with probability $\frac{1}{4}$, independently of the arrival of passengers or previous buses. Just after eight o'clock there is no-one at the bus stop.

(i) What is the probability that the first bus arrives at n minutes past 8?

(ii) If the first bus arrives at 8:05, what is the probability that there are m people waiting for it?

(iii) Each bus can take 25 people and, since it is the terminus, the bus arrive empty. Explain carefully how you would calculate, to two significant figures, the probability that when the first bus arrives it is unable to pick up all the passengers. Your method should need the use of a calculator and standard tables only. There is no need to carry out the calculation.