Section A: Pure Mathematics

1 **(i)** Find the real values of x for which

$$x^3 - 4x^2 - x + 4 \geqslant 0.$$

(ii) Find the three lines in the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 = 0.$$

(iii) On a sketch shade the regions of the (x, y) plane for which

$$x^3 - 4x^2y - xy^2 + 4y^3 \geqslant 0.$$

2 **(i)** Suppose that

$$S = \int \frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x \quad \text{and} \quad T = \int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x.$$

By considering $S + T$ and $S - T$ determine S and T.

(ii) Evaluate $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} (1-4x)\sqrt{\frac{1}{x} - 1}\,\mathrm{d}x$ by using the substitution $x = \sin^2 t$.

3 **(i)** If $\mathrm{f}(r)$ is a function defined for $r = 0, 1, 2, 3, \ldots$, show that

$$\sum_{r=1}^{n} \{\mathrm{f}(r) - \mathrm{f}(r-1)\} = \mathrm{f}(n) - \mathrm{f}(0).$$

(ii) If $\mathrm{f}(r) = r^2(r+1)^2$, evaluate $\mathrm{f}(r) - \mathrm{f}(r-1)$ and hence determine $\displaystyle\sum_{r=1}^{n} r^3$.

(iii) Find the sum of the series $1^3 - 2^3 + 3^3 - 4^3 + \cdots + (2n+1)^3$.

4　By applying de Moivre's theorem to $\cos 5\theta + \mathrm{i}\sin 5\theta$, expanding the result using the binomial theorem, and then equating imaginary parts, show that

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

Use this identity to evaluate $\cos^2\frac{1}{5}\pi$, and deduce that $\cos\frac{1}{5}\pi = \frac{1}{4}(1+\sqrt{5})$.

5　If

$$\mathrm{f}(x) = nx - \binom{n}{2}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} - \cdots + (-1)^{r+1}\binom{n}{r}\frac{x^r}{r} + \cdots + (-1)^{n+1}\frac{x^n}{n}\,,$$

show that

$$\mathrm{f}'(x) = \frac{1-(1-x)^n}{x}\,.$$

Deduce that

$$\mathrm{f}(x) = \int_{1-x}^{1}\frac{1-y^n}{1-y}\,\mathrm{d}y.$$

Hence show that

$$\mathrm{f}(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\,.$$

6　**(i)**　In the differential equation

$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y} = \mathrm{e}^{2x}$$

make the substitution $u = 1/y$, and hence show that the general solution of the original equation is

$$y = \frac{1}{A\mathrm{e}^x - \mathrm{e}^{2x}}\,.$$

(ii)　Use a similar method to solve the equation

$$\frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y^2} = \mathrm{e}^{2x}.$$

7 Let A, B, C be three non-collinear points in the plane. Explain briefly why it is possible to choose an origin equidistant from the three points. Let O be such an origin, let G be the centroid of the triangle ABC, let Q be a point such that $\overrightarrow{GQ} = 2\overrightarrow{OG}$, and let N be the midpoint of OQ.

(i) Show that $\overrightarrow{AQ}$ is perpendicular to $\overrightarrow{BC}$ and deduce that the three altitudes of $\triangle ABC$ are concurrent.

(ii) Show that the midpoints of AQ, BQ and CQ, and the midpoints of the sides of $\triangle ABC$ are all equidistant from N.

[The *centroid* of $\triangle ABC$ is the point G such that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. The *altitudes* of the triangle are the lines through the vertices perpendicular to the opposite sides.]

8 Find functions f, g and h such that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y = \mathrm{h}(x) \qquad (*)$$

is satisfied by all three of the solutions $y = x, y = 1$ and $y = x^{-1}$ for $0 < x < 1$.

If f, g and h are the functions you have found in the first paragraph, what condition must the real numbers a, b and c satisfy in order that

$$y = ax + b + \frac{c}{x}$$

should be a solution of $(*)$?

Section B: Mechanics

9 A particle is projected from a point O with speed $\sqrt{2gh}$, where g is the acceleration due to gravity. Show that it is impossible, whatever the angle of projection, for the particle to reach a point above the parabola

$$x^2 = 4h(h - y),$$

where x is the horizontal distance from O and y is the vertical distance above O. State briefly the simplifying assumptions which this solution requires.

10 A small ball of mass m is suspended in equilibrium by a light elastic string of natural length l and modulus of elasticity λ. Show that the total length of the string in equilibrium is $l(1+mg/\lambda)$.

If the ball is now projected downwards from the equilibrium position with speed u_0, show that the speed v of the ball at distance x below the equilibrium position is given by

$$v^2 + \frac{\lambda}{lm}x^2 = u_0^2.$$

At distance h, where $\lambda h^2 < lmu_0^2$, below the equilibrium position is a horizontal surface on which the ball bounces with a coefficient of restitution e. Show that after one bounce the velocity u_1 at $x = 0$ is given by

$$u_1^2 = e^2u_0^2 + \frac{\lambda}{lm}h^2(1 - e^2),$$

and that after the second bounce the velocity u_2 at $x = 0$ is given by

$$u_2^2 = e^4u_0^2 + \frac{\lambda}{lm}h^2(1 - e^4).$$

11 Two identical uniform cylinders, each of mass m, lie in contact with one another on a horizontal plane and a third identical cylinder rests symmetrically on them in such a way that the axes of the three cylinders are parallel. Assuming that all the surfaces in contact are equally rough, show that the minimum possible coefficient of friction is $2 - \sqrt{3}$.

Section C: Probability and Statistics

12 A school has n pupils, of whom r play hocket, where $n \geqslant r \geqslant 2$. All n pupils are arranged in a row at random.

(i) What is the probability that there is a hockey player at each end of the row?

(ii) What is the probability that all the hockey players are standing together?

(iii) By considering the gaps between the non-hockey-players, find the probability that no two hockey players are standing together, distinguishing between cases when the probability is zero and when it is non-zero.

13 A scientist is checking a sequence of microscope slides for cancerous cells, marking each cancerous cell that she detects with a red dye. The number of cancerous cells on a slide is random and has a Poisson distribution with mean μ. The probability that the scientist spots any one cancerous cell is p, and is independent of the probability that she spots any other one.

(i) Show that the number of cancerous cells which she marks on a single slide has a Poisson distribution of mean $p\mu$.

(ii) Show that the probability Q that the second cancerous cell which she marks is on the kth slide is given by

$$Q = \mathrm{e}^{-\mu p(k-1)} \left\{(1 + k\mu p)(1 - \mathrm{e}^{-\mu p}) - \mu p\right\}.$$

14 **(i)** Find the maximum value of $\sqrt{p(1-p)}$ as p varies between 0 and 1.

(ii) Suppose that a proportion p of the population is female. In order to estimate p we pick a sample of n people at random and find the proportion of them who are female. Find the value of n which ensures that the chance of our estimate of p being more than 0.01 in error is less than 1%.

(iii) Discuss how the required value of n would be affected if (a) p were the proportion of people in the population who are left-handed; (b) p were the proportion of people in the population who are millionaires.